ON HIGHER APPROXIMATION IN THE BOUNDARY-LAYER THEORY

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In [1] Van Dyke applied the method of "outer" and "inner" expansions in powers of $\varepsilon = R^{-1/2}$ to the problem of finding the unseparated laminar flow of a viscous incompressible fluid at high Reynolds number R past a semi-infinite body, and investigated in detail the first two terms of those expansions. In the present work we consider terms of higher order. In the examples of flow in a diffusor near the stagnation point, and so on, it is shown that the "outer" solution for such a problem consists of a part that is represented by an asymptotic power series in ε, and a part that is not represented by a series of that sort (it is $O(e^{-a/\varepsilon})$, $a > 0$). In the example of flow in a diffusor, for which there is an exact solution, it is found that the basic part of the solution, represented by a power series in ε, can be found in the usual way independently of the "exponential part", which can then be found by a perturbation method. A method is proposed for joining the "outer" and "inner" solutions that is applicable to the problem of flow past a body at high R.

1. We consider convergent laminar flow of a viscous incompressible fluid with constant coefficient of kinematic viscosity ν at high Reynolds R in a plane diffusor using a system of polar coordinates r, θ (see Fig.1).

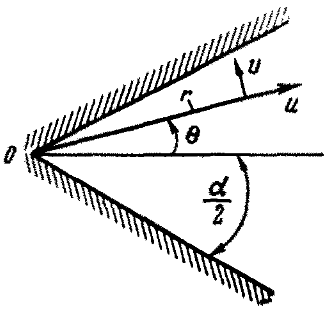

Fig. 1

We denote the half-angle of the diffusor by $\frac{1}{2}\alpha$, and the velocity components in the directions of increasing r and θ by u and v, respectively. It is well known (see, for example, [2]) that in this flow $v = 0$ and $u = r^{-1}V(\theta)$. We express the function $V(\theta)$ in the form $V(\theta) = AU(\theta)$, where A is a constant with dimensions L^2T^{-1}, and the dimensionless function $U(\theta)$ satisfies Equation

$$\varepsilon^2 (U'' + 4U) + U^2 - 1 = 0$$
$$(\varepsilon = R^{-1/2},\ R = A/\nu) \qquad (1.1)$$

Here R is the Reynolds number, and primes indicate differentiation with respect to θ

The boundary conditions for U are

$$U(\pm {}^1/_2\alpha) = 0 \quad \text{or} \quad U({}^1/_2\,\alpha) = 0, \quad U'(0) = 0 \tag{1.2}$$

We will solve the problem by finding $U(\theta)$ with the method of "outer" and "inner" expansions. Outside the boundary layer $U \to \pm 1$ as $\varepsilon \to 0$, as is evident form Equation (1.1); we consider converging flow, $U \to -1$.

We seek the outer solution in the form

$$U = U_0 + o(\varepsilon^m) = -1 + \varepsilon^2 U_{02} + \varepsilon^4 U_{04} + \ldots + o(\varepsilon^m) \tag{1.3}$$

(m is an arbitrary positive integer)

Substituting (1.3) into (1.1) and equating like powers of ε, we see that all U_{0k} = const ; therefore setting $U_0'' = 0$ we find from (1.1)

$$U_0^2 + 4\varepsilon^2 U_0 - 1 = 0, \quad \text{or} \quad U_0 = -\sqrt{1 + 4\varepsilon^4} - 2\varepsilon^2 = -1 - 2\varepsilon^2 - 2\varepsilon^4 + \ldots \tag{1.4}$$

We seek an "inner" solution in the boundary layer of the form

$$U = u_0(\vartheta) + \varepsilon^2 u_2(\vartheta) + \varepsilon^4 u_4(\vartheta) + \ldots \qquad (\vartheta = ({}^1/_2\alpha - \theta)\,\varepsilon^{-1}) \tag{1.5}$$

We transform (1.1) to the variable ϑ, $d^2U / d\vartheta^2 + U^2 - 1 + 4\varepsilon^2 U = 0$; and substituting (1.5) and equating coefficients of like powers of ε we obtain

$$d^2u_0 / d\vartheta^2 + u_0^2 - 1 = 0, \qquad d^2u_2 / d\vartheta^2 + 2u_0(u_2 + 2) = 0 \ \text{etc.} \tag{1.6}$$

The boundary conditions of no slip at the wall and the conditions of joining the "inner" and "outer" solutions are, according to (1.4),

$$u_0(0) = 0, \quad u_0(\infty) = -1; \quad u_2(0) = 0, \quad u_2(\infty) = -2 \ \text{etc.} \tag{1.7}$$

The function $u_0(\vartheta)$ is found from (1.6) and (1.7) in closed form as

$$u_0(\vartheta) = -1 + 12\,[(\sqrt{3} + \sqrt{2})\, e^{{}^1/_2 \sqrt{2}\vartheta} + (\sqrt{3} - \sqrt{2})\, e^{-{}^1/_2 \sqrt{2}\vartheta}]^{-2} \tag{1.8}$$

(This is a known solution in boundary-layer theory).

The function $u_2(\vartheta)$ is determined uniquely by (1.6) and (1.7) as

$$u_2(\vartheta) = -2 + O(\vartheta e^{-\sqrt{2}\vartheta}) \qquad \text{for } \vartheta \to \infty \tag{1.9}$$

We now find U from the "inner" solution for large ϑ. From (1.4), (1.5), (1.8) and (1.9) we obtain

$$U = -1 - 2\varepsilon^2 + \ldots + \frac{12}{(\sqrt{3} + \sqrt{2})^2} \exp(-\sqrt{2}\vartheta) + \ldots =$$

$$= -\sqrt{1 + 4\varepsilon^4} - 2\varepsilon^2 + \frac{12}{(\sqrt{3} + \sqrt{2})^2} \exp\left[-\frac{\sqrt{2}}{\varepsilon}\left(\frac{\alpha}{2} - \theta\right)\right] + \ldots =$$

$$= U_0 + \frac{12}{(\sqrt{3} + \sqrt{2})^2} \exp\left[-\frac{\sqrt{2}}{\varepsilon}\left(\frac{\alpha}{2} - \theta\right)\right] + \ldots \tag{1.10}$$

Equation (1.10) clearly shows that there should be present in the "outer" solution exponential terms that are not represented by an asymptotic series in powers of ε. We find the dominant one by a perturbation method. We seek an outer solution in the form $U = U_0 + U_1$. After substituting U into (1.1) we obtain

$$\varepsilon^2 (U_1'' + 4U_1) + 2U_0 U_1 + U_1^2 = 0 \tag{1.11}$$

Discarding U_1^2 in (1.11) and solving the resulting equation, we obtain the leading term of U_1 in the form

$$U_1 = c\{\exp[(\theta - {}^1/_2\alpha)\sqrt{2}\,\varepsilon^{-1}(1+4\varepsilon^4)^{1/4}] + \exp[-(\theta + {}^1/_2\alpha)\sqrt{2}\,\varepsilon^{-1}(1+4\varepsilon^4)^{1/4}]\} =$$
$$= c\exp[-({}^1/_2\alpha - \theta)\sqrt{2}\,\varepsilon^{-1}(1+4\varepsilon^4)^{1/4}]\{1 + \exp[-\alpha\varepsilon^{-1}\sqrt{2}(1+4\varepsilon^4)^{1/4}]\}$$
$$(c = 12(\sqrt{3}+\sqrt{2})^{-2}) \qquad (1.12)$$

We find the constant by equating the leading term of $U - U_0$ from (1.10) with the leading term of U_1 according to (1.12). Finally we write the "outer" solution in the form

$$U = -\sqrt{1+4\varepsilon^4} - 2\varepsilon^2 + \frac{12}{(\sqrt{3}+\sqrt{2})^2}\exp[-\vartheta\sqrt{2}(1+4\varepsilon^4)^{1/4}]\times$$
$$\times\left\{1 + \exp\left[-\frac{\alpha\sqrt{2}}{\varepsilon}(1+4\varepsilon^4)^{1/4}\right]\right\} + \ldots \qquad (1.13)$$

(The dots indicate terms of higher order in ε than those shown). As a check of (1.13) we equate U_1 as given by (1.12) with the U_1 obtained from the exact solution on the axis of the diffusor (at $\theta = 0$). If we set $U_1(0) = U_1^*$, integrating (1.11) and satisfying the boundary conditions $U_1(\alpha/2) = -U_0$ (the condition of no slip) and $U'(0) = 0$, we obtain the exact equation

$$\int_{U_1^*}^{U^\circ} \frac{d\tau}{\sqrt{{}^2/_3(U_1^{*3} - \tau^3) + [2(1+4\varepsilon^4)^{1/2}(\tau^2 - U_1^{*2})]}} = \frac{\alpha}{2\varepsilon} \quad (U^\circ = \sqrt{1+4\varepsilon^4} + 2\varepsilon^2) \quad (1.14)$$

From (1.12) we obtain

$$U_1^* = \frac{24}{(\sqrt{3}+\sqrt{2})^2}\exp\left[-\frac{\alpha}{\varepsilon\sqrt{2}}(1+4\varepsilon^4)^{1/4}\right] + \ldots \approx$$
$$\approx 2.41\exp\left[-\frac{\alpha}{\varepsilon\sqrt{2}}(1+4\varepsilon^4)^{1/4}\right] + \ldots \qquad (1.15)$$

Taking the expression in square brackets out from under the radical in the denominator of the integral in (1.14) and expanding the remaining expression in a series in powers of

$$-{}^1/_3(1+4\varepsilon^4)^{-1/2}\frac{U_1^{*3} - \tau^3}{U_1^{*2} - \tau^2}$$

we obtain after integration:

keeping one term in the series

$$U_1^* \approx 2\exp\left[-\frac{\alpha}{\varepsilon\sqrt{2}}(1+4\varepsilon^4)^{1/4}\right] + \ldots$$

keeping two terms

$$U_1^* \approx 2.36\exp\left[-\frac{\alpha}{\varepsilon\sqrt{2}}(1+4\varepsilon^4)^{1/4}\right] + \ldots \qquad (1.16)$$

Comparing (1.15) and (1.16) we are convinced of their identity. (The numerical coefficients preceding the exponent should agree if all terms of the above-mentioned series were considered).

From this example the following conclusions can be drawn. The outer solution consists of two components: a part represented by an asymptotic series in powers of ε, and a part not represented in that form. The basic "power part" of the solution can be obtained by the method of "inner" and "outer" expansions, operating as if the "exponential part" did not exist, after which that part of the solution can be found by a perturbation method. The presence of an "exponential part" of the solution is a general feature of solutions of the Navier-Stokes equations. This property can be observed in the examples of flow near a stagnation point of the flow, near a plate, in the problem of diffusion of a vortex, in the one-dimensional problem of a shock wave in the case of a perfect gas with constant transport coefficients, and so on.

2. Consider, for example, the problem of flow of a viscous fluid in the vicinity of a stagnation point in the plane or axisymmetric case [1]. Van Dyke [1] uses the conventional coordinate system employed in boundary-layer theory: n is the normal distance from the surface to the point under consideration, and s the distance from that normal to the stagnation point along the arc of the contour.

The "outer" and "inner" expansions for the stream function ψ are taken in the form [1]

$$\psi \sim \Psi_1(s, n) + \varepsilon\Psi_2(s, n) + \ldots, \qquad \varepsilon = R^{-1/2}$$

Here R is the Reynolds number formed with the radius of curvature of the nose of the body.

$$\psi \sim \varepsilon\psi_1(s, N) + \varepsilon^2\psi_2(s, N) + \ldots, \qquad N = n\varepsilon^{-1} \tag{2.1}$$

In the vicinity of the stagnation point in the axisymmetric case

$$\psi_1(s, N) = (^1/_2\, U_{11})^{1/2} s^2 [f(\eta) + O(s^2)], \qquad \eta = (2U_{11})^{1/2} N \tag{2.2}$$

where U_{11} is determined by the series expansion for the velocity U_1 of an ideal fluid on the contour

$$U_1(s, 0) = U_{11}s + O(s^3)$$

In the plane case

$$\psi_1(s, N) = U_{11}{}^{1/2} s [f(\eta) + O(s)], \qquad \eta = U_{11}{}^{1/2} N \tag{2.3}$$

The equation satisfied by f is

$$f''' + ff'' = \beta(f'^2 - 1), \qquad f(0) = f'(0) = 0, \ \ f'(\infty) = 1 \tag{2.4}$$

Here $\beta = 1$ for plane flow and $\beta = \frac{1}{2}$ for the axisymmetric case. (Primes indicate differentiation with respect to η).

We investigate the behavior of f for large η. It is established by numerical integration [1] that $f \sim \eta - \beta_1$ for large η (where $\beta_1 = 0.80455$ for the axisymmetric and $\beta_1 = 0.647900$ for the plane case). We represent f in the form

$$f = \eta - \beta_1 + \varphi \qquad (\varphi(\infty) = \varphi'(\infty) = 0) \tag{2.5}$$

We determine the principal term in φ for $\eta \to \infty$. We put (2.4) into the form [1]

$$(1 + \beta) f'^2 = (f'' + ff' + \beta\eta)'$$

Substituting (2.5), we obtain

$$(1 + \beta)(2\varphi' + \varphi'^2) = [\varphi'' + \varphi + (\eta - \beta_1)\varphi' + \varphi\varphi']' \tag{2.6}$$

To determine the principal term in φ we naturally neglect in (2.6) small quantities compared with the remaining terms

$$2(1 + \beta)\varphi' = [\varphi'' + \varphi + (\eta - \beta_1)\varphi']'$$

Hence, after a single integration, we have

$$\varphi'' + (\eta - \beta_1)\varphi' - (1 + 2\beta)\varphi = c$$

The constant $c = 0$, because it corresponds to the particular solution $\varphi = -c/(1 + 2\beta)$, which does not tend to zero as $\eta \to \infty$. Setting $\eta - \beta_1 = \xi$, we have

$$\varphi'' + \xi\varphi' - m\varphi = 0, \qquad m = 1 + 2\beta \tag{2.7}$$

Since $m = 2$ and 3, respectively for the axisymmetric and plane cases, one solution φ_1 of (2.7) is expressed in terms of the Chebyshev-Hermite polynomial

$$\varphi_1 = H_m\left(\frac{i\xi}{\sqrt{2}}\right), \qquad \varphi_1 = 1 + \xi^2, \quad m = 2; \qquad \varphi_1 = \xi^3 - 3\xi, \quad m = 3 \tag{2.8}$$

A second linearly independent solution φ_2 of (2.7) is determined by Equation

$$\varphi_2 = \varphi_1 \int_{\xi}^{\infty} \exp \frac{-t^2}{2} \frac{dt}{\varphi_1^2}$$

Hence, integrating by parts, it is easy to find the asymptotic formula for φ_2 as $\xi \to \infty$

$$\varphi_2 \sim \xi^{-3} e^{-1/2\xi^2}, \quad m = 2; \qquad \varphi_2 \sim \xi^{-4} e^{-1/2\xi^2}, \quad m = 3 \tag{2.9}$$

From (2.8) and (2.9)

$$\varphi \sim c\xi^{-(m+1)} e^{-1/ \xi^2} \tag{2.10}$$

(where the constant c depends upon m). From (2.5) and (2.10) we obtain

$$f = \xi + c\xi^{-(m+1)} e^{-1/2\xi^2} + \ldots \tag{2.11}$$

(The dots indicate terms of higher order for $\xi \to \infty$). We recall that

$$\xi = \eta - \beta_1 = U_{11}^{1/2} N - \beta_1 = U_{11}^{1/2} n\varepsilon^{-1} - \beta_1$$

so that for large N in the conversion to the variables n and s of the "outer" expansion it follows from (2.1), (2.2), (2.3) and (2.11) that the "outer" solution contains an "exponential part".

3. In the work of Van Dyke [1] the first two terms of the "outer" and "inner" expansions were joined by using a general method of joining asymptotic expansions proposed by Lagerstrom [3]. We use below a method of joining these expansions that arises from physical considerations for the problem under consideration. We are henceforth interested only in the "power part" of the solution, and use the notation of [1]. The "outer" expansion for the component v in the direction of increasing n is taken in the form

$$v(s, n, R) \sim V_1 (s, n) + \varepsilon V_2 (s, n) + \varepsilon^2 V_3 (s, n) + \ldots, \ \varepsilon = R^{-1/2} \tag{3.1}$$

(R is the Reynolds number)

The other quantities are represented analogously. Upon substituting these expansions into the Navier-Stokes equations and the continuity equation, and equating the coefficients of like powers of ε , one obtains a system of equations for the coefficients of the expansions that have the type of the Euler equations. The viscous terms in these equations are known functions of the coefficients with smaller indices, and can be treated as mass forces in the Euler equations. For this reason in determining the coefficients in the expansion (3.1) it is necessary to know only the behavior of the $V_k(s,n)$ for $n \to 0$. The expansion (3.1) is valid outside the boundary layer, but each term of such an expansion can be continued analytically to the surface of the body, $n = 0$, and here it is necessary to substitute a corresponding condition for it. We turn to the "inner" expansion; it is taken in the form

$$v (s, n, R) \sim \varepsilon v_1 (s, N) + \varepsilon^2 v_2 (s, N) + \varepsilon^3 v_3 (s, N) + \ldots, \qquad N = n\varepsilon^{-1} \tag{3.2}$$

The other quantities are represented analogously. We assume that the regions of validity of (3.1) and (3.2) overlap [1]. For large N but small n all terms of (3.2) are of the same order, $O(1)$, and it is necessary to regroup them, transforming to the variables n, s and discarding exponential terms, after which the expansion (3.1) should be obtained. (The expansions (3.1) and (3.2) represent one and the same solution). Because the coefficients of the "outer" expansion are obtained from equations of Euler type, there is no reason to suppose that $V_k(s,n) \to \infty$ as $n \to 0$; consequently one expects for $v_k(s,N)$ as $N \to \infty$ the behavior

$$v_k (s, N) \sim a_{k0} (s) N^k + a_{k1} (s) N^{k-1} + \ldots + a_{kk} (s) + [\exp] \tag{3.3}$$

Here [exp] stands for exponential terms. This sort of behavior of the $v_k(s,N)$ is confirmed by consideration of the first two terms of (3.2) found by Van Dyke [1]. For large N but small n we obtain from (3.2), (3.3) and (3.)

$$v(s,n,R)\sim\sum_{k=1}^{\infty}\varepsilon^k v_k(s,N)\sim\sum_{k=1}^{\infty}\varepsilon^k\sum_{l=0}^{k}a_{kl}N^{k-l}\sim\sum_{k=1}^{\infty}\sum_{l=0}^{k}a_{kl}n^{k-l}\varepsilon^l\sim\sum_{l=1}^{\infty}a_{l0}n^l+$$
$$+\sum_{l=1}^{\infty}\varepsilon^l\sum_{k=l}^{\infty}a_{kl}n^{k-l}\sim\sum_{r=1}^{\infty}\varepsilon^{r-1}V_r(s,n)\qquad(3.4)$$

It follows from (3.4) that as $n\to 0$

$$V_1(s,n)\to 0,\qquad V_r(s,n)\to a_{r-1,\,r-1}(s)\qquad(3.5)$$

The solution of the problem is now carried out in the following order. The first term of the expansion (3.1) is sought with the boundary condition $V_1(s,0)=0$. (This is the solution of the problem of flow past the contour of a stream of ideal fluid). Then the first term of the expansion (3.2) is found. (This is the solution of boundary-layer theory). Then $a_{11}(s)$ is determined (see (3.3)) and the problem for the second term of the expansion (3.1) is solved with the boundary condition $V_2(s,0)=a_{11}(s)$ (see (3.5)). After that the second term of the expansion (3.2) is found. (We do not reproduce the matching conditions on the solution for $N\to\infty$; see [1]). Then $a_{22}(s)$ is found from (3.3) and the third term in (3.1) is determined, whereupon the third term in (3.2) is determined, and so on. For the coefficients of the "inner" expansion it is necessary to display as many conditions for $N\to\infty$ as are needed for their unique determination; for the pressure p and the component u in the direction of increasing s , these conditions are obtained from the "outer" expansion for small n after transformation from n to N . We find, for example, the conditions for u . The "outer" expansion for u has the form

$$u(s,n,R)\sim U_1(s,n)+\varepsilon U_2(s,n)+\varepsilon^2U_3(s,n)+\ldots\qquad(3.6)$$

Because

$$U_k(s,n)\sim\sum_{l=0}^{\infty}U_{kl}(s)\,n^l\qquad\text{for }n=0$$

we obtain from (3.6), upon transforming from n to $N=n\varepsilon^{-1}$

$$u(s,n,R)\sim\sum_{r=0}^{\infty}U_{r+1}(s,n)\,\varepsilon^r\sim\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}U_{r+1,\,l}(s)\,\varepsilon^{r+l}N^l\sim$$
$$\sim\sum_{k=0}^{\infty}\varepsilon^k\sum_{r=0}^{k}U_{r+1,\,k-r}(s)\,N^{k-r}\sim\sum_{k=0}^{\infty}\varepsilon^k u_{k+1}(s,N)\qquad(3.7)$$

From (3.7) we obtain for $N\to\infty$

$$u_{k+1}(s,N)\sim\sum_{r=0}^{k}U_{r+1,\,k-r}(s)\,N^{k-r}\qquad(3.8)$$

In particular, it follows from (3.8) that for $N\to\infty$

$$u_1(s,N)=U_{10}(s),\qquad u_2(s,N)\sim U_{11}(s)\,N+U_{20}(s)$$
$$u_3(s,N)\sim U_{12}(s)\,N^2+U_{21}(s)\,N+U_{30}(s)\ \text{etc.}$$

Analogous results hold for the pressure p .

The foregoing is confirmed by consideration of the first two terms of the "inner" expansions obtained by Van Dyke [1].

We remark in conclusion that the "exponential" terms will play a role in the investigation of flow at moderate Reynolds number, which turns out to be of the order of 10 for the flat plate [4].

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